M.Sc.

Equivalent Norms: — let a linear space L be made into a normed linear space in two ways and let the two norms of a vector x in L be denoted by $\|x\|_1$ and $\|x\|_2$. Then these norms are said to be equivalent, written $\| \ \|_1 \sim \| \ \|_2$ iff they generate the same topology on L.

Theorem (A):- let N be a Normed linear space and suppose two norms $\| \ \|_1$ and $\| \ \|_2$ are defined on N. Then these norms are equivalent if and only if there exist positive real numbers m and M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for every x in N.

Proof: — let N_i be the normed linear space with its norm $\| \ \|_i$ $(i = 1, 2)$. let $T(x) = x$ and consider T as a linear transformation with domain N_1 and range N_2. Then T^{-1} is a linear transformation with domain N_2 and range N_1 such that

$$T(x) = x \Rightarrow T^{-1}(x) = x.$$

Now T is continuous $\Rightarrow$ T is bounded

$\Rightarrow \exists$ positive number M such that

$\|T(x)\|_2 \leq M\|x\|_1 \ \forall x \in N$

$\Rightarrow \|x\|_2 \leq M\|x\|_1 \ \forall x \in N$ —— (1)

$(\because T(x) = x)$

Again T^{-1} is continuous $\Rightarrow T^{-1}$ is bounded

$\Rightarrow \exists$ positive number K such that

$\|T^{-1}(x)\|_1 \leq K\|x\|_2 \ \forall x \in N_2.$

$\Rightarrow \|x\|_1 \le K\|x\|_2$ $(\because T^{-1}(x) = x)$

$\Rightarrow \frac{1}{K}\|x\|_1 \le \|x\|_2$ $(\because K > 0)$

$\Rightarrow m\|x\|_1 \le \|x\|_2$, setting $\frac{1}{K} = m$ ——— (2)

Also T and T^{-1} are continuous

$\Rightarrow$ inverse image of open sets in N_2 and N_1 under T and T^{-1} respectively are open in N_1 and N_2

$\Rightarrow$ open sets in N_1 and N_2 are the same

$(\because T$ and T^{-1} are identity transformations$)$

$\Rightarrow \| \ \|_1$ and $\| \ \|_2$ induce the same topology on N. ——— (3)

From (1), (2), (3) and the above definition we conclude that $\| \ \|_1$ and $\| \ \|_2$ are equivalent.

$\Rightarrow$ there exist positive numbers m and M such that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1 \ \forall x \in N$.

~~Refer for further discussion of equivalent norms~~

Theorem (B) :- Let N and N' be normed linear spaces and let $T: A \to N'$ be any linear transformation. If N is finite-dimensional then T is continuous (or bounded).

Proof :- Let $\dim N = n$ and $\{e_1, e_2, \ldots, e_n\}$ be a basis for N. Then to each $x \in N$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

Since T is linear, we have

$$T(x) = \sum_{i=1}^{n} \alpha_i T(e_i) \quad \text{——— (1)}$$

Now, Consider zeroth norm on N defined by

$$\|x\|_0 = \max|\alpha_i| \quad \text{——— (2)}$$

If $\| \ \|$ is the norm on N', then (1) gives

$$\|T(x)\| = \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \leq \sum_{i=1}^{n} |\alpha_i| \, \|e_i\|$$

$$\leq \|x\|_0 \sum_{i=1}^{n} \|e_i\| \qquad (3)$$

Since the basis is fixed $\sum_{i=1}^{n} \|e_i\|$ is a positive Constant
Hence letting $M = \sum_{i=1}^{n} \|e_i\|$, we have

$$\|T(x)\| \leq M \|x\|_0$$

Thus T is bounded and hence it is Continuous.

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